October Math Gems

PROBLEM OF THE WEEK 8

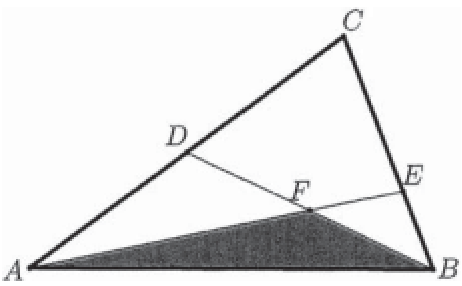
§1 problems

Problem 1.1. Let a, b, c be integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$ prove that abc is a perfect cube using AM-GM inequality!

Problem 1.2. Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 3, \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

Problem 1.3. Given that ABC is a triangle with D being the midpoint of AC and E a point on CB such that $CE = 2EB$. If AE and BD intersect at point F and the area of triangle AFB is equal to 1 unit, find the area of ABC.

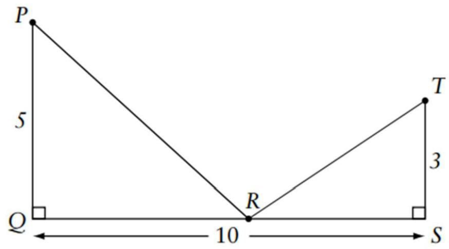


Problem 1.4. If $\tan a = \frac{1}{2m+1}$,and $\tan b = \frac{m}{m+m+1}$ where a, b are acute, then $a + b =$

Problem 1.5. If $f(x) = (a - x^n)^{\frac{1}{n}}$. Prove that $f(x) = f^{-1}(x)$

Problem 1.6. Two integers have a sum of 26. When two more integers are added to the first two integers the sum is 41. Finally when two more integers are added to the sum of the previous four integers the sum is 57. What is the minimum number of even integers among the 6 integers?

Problem 1.7. A student wishes to move from point P to point T via point R on line QS. What is the distance QR that makes the total distance traveled minimum?



Problem 1.8. Simplify the give expression

$$\log_2 3 \times \log_3 4 \times \log_4 5 \times ... \times \log_{127} 128$$

Problem 1.9. In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16}, \qquad \cos B = \frac{7}{8}, \qquad \text{and} \qquad \cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

Problem 1.10. be the system

$$\begin{cases} 3x_1^2 + 3x_2^2 + 3x_3^2 = 6x_4 - 1 \\ 3x_1^2 + 3x_2^2 + 3x_4^2 = 6x_3 - 1 \\ 3x_1^2 + 3x_3^2 + 3x_4^2 = 6x_2 - 1 \\ 3x_2^2 + 3x_3^2 + 3x_4^2 = 6x_1 - 1 \end{cases}$$

Calculate the value the

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$$

Problem 1.11. $\frac{1+\sin a - \cos 2a - \sin 3a}{2\sin^2 a + \sin a - 1}$ Simpify expression

Problem 1.12. If

$$\sum_{r=0}^{n-1} \log_2 \left(\frac{r+2}{r+1} \right) = \prod_{r=10}^{99} \log_r (r+1)$$

What is the value of n

Problem 1.13. Known that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

The minimum value of

$$3^{\sin^6(x)} + 3^{\cos^6(x)}$$

can be written in the form of ab^c. Find $4ac + b$

Problem 1.14. Let m, n be positive integers with $m < n$. Find the a closed form for the sum

$$\frac{1}{\sqrt{m} + \sqrt{m+1}} + \frac{1}{\sqrt{m+1} + \sqrt{m+2}} + \cdots + \frac{1}{\sqrt{n-1} + \sqrt{n}}$$

Problem 1.15. Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$

Problem 1.16. Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right) + 12\cos a = 5\cos^4\left(\frac{a}{2}\right) + 12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a = \frac{m}{n}$. Find $10m + n$.

Problem 1.17. Let z be a complex number that satisfies the equation

$$\frac{z-4}{z^2-5z+1} + \frac{2z-4}{2z^2-5z+1} + \frac{z-2}{z^2-3z+1} = \frac{3}{z}.$$

Over all possible values of z, find the sum of the values of

$$\left|\frac{1}{z^2-5z+1} + \frac{1}{2z^2-5z+1} + \frac{1}{z^2-3z+1}\right|.$$

Problem 1.18. Let ABC be an acute triangle with $\angle ABC = 60°$. Suppose points D and E are on lines AB and CB, respectively, such that CDB and AEB are equilateral triangles. Given that the positive difference between the perimeters of CDB and AEB is 60 and $DE = 45$, what is the value of $AB \cdot BC$?

Problem 1.19. Let $f(x)$ be a quadratic with integer coefficients. Suppose there exist positive primes $p < q$ such that $f(p) = f(q) = 87$ and $f(p+q) = 178$. Find $p^2 + q^2$.

Problem 1.20. Find all pairs (x, y) of real numbers satisfying the system :

$$\begin{cases} x + y = 3 \\ x^4 - y^4 = 8x - y \end{cases}$$